



# A Portrayal of Integer Solutions to Non-homogeneous Ternary Higher Degree Diophantine Equation

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**ABSTRACT:** This paper aims at determining non-zero distinct integer solutions to Ternary non-homogeneous higher degree Diophantine equation given by  $ax^2 + by^2 = (a+b)(c^2 + ab)z^{2s+1}$ ,  $a, b > 0$  & square-free. Varieties of integer solutions are obtained through utilizing substitution technique and factorization method.

$$ax^2 + by^2 = (a+b)(c^2 + ab)z^{2s+1}, a, b > 0 \text{ \& square-free}$$

**KEYWORDS:** Non-homogeneous equation, Ternary higher degree equation, Integer solutions, Substitution technique, Factorization method

## I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous and non-homogeneous equations of higher degree have aroused the interest of many mathematicians since antiquity. For example, in [1-5], the authors are analysed cubic equations with multiple variables. In [6, 7], the quartic and quintic equations have been considered for obtaining many integer solutions. In [8-10], the sixth degree equations with multiple variables have been discussed for various choices of integer solutions. In [11], a Heptic equation with three unknowns has been studied for finding its integer solutions. In [12], an octic equation has been presented with different sets of integer solutions. In [13,16,17], a ninth degree equation with three unknowns has been studied for finding its solutions in integers. In [14,15,18,19], polynomial equations of degree ten with six unknowns have been searched for determining their integer solutions. The above problems motivated us to get integer solutions for another interesting non-homogeneous higher degree equation with five unknowns given by  $ax^2 + by^2 = (a+b)(c^2 + ab)z^{2s+1}$ . Substitution technique and factorization method are utilized to determine the same.

## II. METHOD OF ANALYSIS

The non-homogeneous ternary higher degree equation to be solved is

$$ax^2 + by^2 = (a+b)(c^2 + ab)z^{2s+1} \quad (1)$$

where the product  $a b > 0$  and square-free.

Introduction of the transformations

$$x = X + b T, y = X - a T \tag{2}$$

in (1) leads to the non-homogeneous ternary higher degree equation

$$X^2 + a b T^2 = (c^2 + a b) z^{2s+1} \tag{3}$$

The process of getting varieties of integer solutions to (1) through solving (3) is presented below:

### Process 1

Choosing

$$X = c T \tag{4}$$

in (3), one obtains  $T^2 = z^{2s+1}$

which is satisfied by

$$\begin{aligned} T &= \alpha^{2s+1} \\ z &= \alpha^2 \end{aligned} \tag{5}$$

Using (5) in (4), we have

$$X = c \alpha^{2s+1} \tag{6}$$

In view of (2), one has

$$\begin{aligned} x &= (c + b) \alpha^{2s+1}, \\ y &= (c - a) \alpha^{2s+1} \end{aligned} \tag{7}$$

Thus, the values of  $x, y, z$  given by (7) & (5) satisfy (1).

### Process 2

The choice

$$T = z^s k \tag{8}$$

in (3) gives

$$X^2 = z^{2s} [(c^2 + a b) z - a b k^2] \tag{9}$$

which is satisfied by

$$z = k^2 + 2nck + (c^2 + ab)n^2 \tag{10}$$

$$X = [k^2 + 2nck + (c^2 + ab)n^2]^s [ck + (c^2 + ab)n]$$

In view of (8), we have

$$T = [k^2 + 2nck + (c^2 + ab)n^2]^s k \tag{11}$$

Using (10) & (11) in (2), we get

$$x = [k^2 + 2nck + (c^2 + ab)n^2]^s [(c + b)k + (c^2 + ab)n], \tag{12}$$

$$y = [k^2 + 2nck + (c^2 + ab)n^2]^s [(c - a)k + (c^2 + ab)n],$$

Thus, the values of x, y, z given by (12) & (10) satisfy (1).

**Process 3**

Assume

$$z = \alpha^2 + ab\beta^2 \tag{13}$$

Consider the integer  $(c^2 + ab)$  on the R.H.S. of (3) as

$$c^2 + ab = (c + i\sqrt{ab})(c - i\sqrt{ab}) \tag{14}$$

Substitute (13) & (14) in (3). Utilizing the method of factorization and equating the positive factors, we have

$$X + i\sqrt{ab}T = (c + i\sqrt{ab})(\alpha + i\sqrt{ab}\beta)^{2s+1} \tag{15}$$

$$= (c + i\sqrt{ab})[f(\alpha, \beta) + i\sqrt{ab}g(\alpha, \beta)]$$

where

$$f(\alpha, \beta) = \frac{1}{2}[(\alpha + i\sqrt{ab}\beta)^{2s+1} + (\alpha - i\sqrt{ab}\beta)^{2s+1}] \tag{16}$$

$$g(\alpha, \beta) = \frac{1}{2i\sqrt{ab}}[(\alpha + i\sqrt{ab}\beta)^{2s+1} - (\alpha - i\sqrt{ab}\beta)^{2s+1}]$$

On equating the real and imaginary parts in (15), we have

$$X = cf(\alpha, \beta) - abg(\alpha, \beta)$$

$$T = f(\alpha, \beta) + cg(\alpha, \beta)$$

In view of (2), we get

$$x = (c + b)f(\alpha, \beta) + b(c - a)g(\alpha, \beta), \tag{17}$$

$$y = (c - a)f(\alpha, \beta) - a(c + b)g(\alpha, \beta)$$

Thus , (17) & (13) satisfy (1).

**Process 4**

Write (3) as

$$X^2 + a b T^2 = (c^2 + a b) z^{2s+1} * 1 \tag{18}$$

Assume

$$z = (m^2 + a b n^2)^2 (\alpha^2 + a b \beta^2) \tag{19}$$

Consider the integer 1 on the R.H.S. of (18) as

$$1 = \frac{(m^2 - a b n^2 + i\sqrt{a b} 2 m n) (m^2 - a b n^2 - i\sqrt{a b} 2 m n)}{(m^2 + a b n^2)^2} \tag{20}$$

Substitute (14), (19) ,(20) in (18). Employing the factorization method and equating the coefficients of corresponding terms ,we have

$$\begin{aligned} X + i\sqrt{a b} T &= (c + i\sqrt{a b}) (m^2 + a b n^2)^{2s+1} (f(\alpha, \beta) + i\sqrt{a b} g(\alpha, \beta)) \frac{(m^2 - a b n^2 + i\sqrt{a b} 2 m n)}{(m^2 + a b n^2)} \\ &= (m^2 + a b n^2)^{2s} (f(\alpha, \beta) + i\sqrt{a b} g(\alpha, \beta)) \{ [c(m^2 - a b n^2) - 2 a b m n] + \\ &\quad i\sqrt{a b} [ (m^2 - a b n^2) + 2 c m n] \} \end{aligned}$$

On equating the real and imaginary parts , we get

$$\begin{aligned} X &= (m^2 + a b n^2)^{2s} \{ f(\alpha, \beta) [c(m^2 - a b n^2) - 2 a b m n] - a b g(\alpha, \beta) [ (m^2 - a b n^2) + 2 c m n] \} \\ T &= (m^2 + a b n^2)^{2s} \{ f(\alpha, \beta) [ (m^2 - a b n^2) + 2 c m n] + g(\alpha, \beta) [c(m^2 - a b n^2) - 2 a b m n] \} \end{aligned}$$

From (2) , one has

$$\begin{aligned} x &= (m^2 + a b n^2)^{2s} \{ f(\alpha, \beta) [(c + b)(m^2 - a b n^2) + 2 b(c - a) m n] + \\ &\quad g(\alpha, \beta) [b(c - a)(m^2 - a b n^2) - 2 a b(b + c) m n] \}, \\ y &= (m^2 + a b n^2)^{2s} \{ f(\alpha, \beta) [(c - a)(m^2 - a b n^2) - 2 a(b + c) m n] - \\ &\quad g(\alpha, \beta) [a(b + c)(m^2 - a b n^2) + 2 a b(c - a) m n] \}, \end{aligned} \tag{21}$$

Thus ,(19) & (21) satisfy (1).

**Process 5**

The insertion of the transformations

$$X = (c^2 + a b)^{s+1} \alpha^s P, T = (c^2 + a b)^{s+1} \alpha^s Q, z = (c^2 + a b) \alpha \tag{22}$$

in (3) leads to

$$P^2 + abQ^2 = \alpha \tag{23}$$

If  $\alpha$  is of the form  $(A^2 + abQ^2)^n$ , then, we get

$$P = \frac{1}{2} [(A + i\sqrt{ab}B)^n + (A - i\sqrt{ab}B)^n]$$

$$Q = \frac{1}{2i\sqrt{ab}} [(A + i\sqrt{ab}B)^n - (A - i\sqrt{ab}B)^n]$$

Substituting the above values of  $\alpha, P, Q$  in (22) and using (2), the corresponding integer solutions to (1) are obtained.

### III. CONCLUSION

In this paper, an attempt has been made to obtain many integer solutions to non-homogeneous Diophantine equation of higher degree with three unknowns. Substitution strategy and factorization technique are utilized for finding the required integer solutions to the higher degree equation with three unknowns. In this analysis, the given equation is reduced to ternary higher degree equation for which the integer solutions can be found elegantly.

### REFERENCES

- [1] S.Vidhyalakshmi, J.Shanthi and M.A.Gopalan, Observation on the paper entitled Integral solution of the Homogeneous ternary cubic equation  $x^3 + y^3 = 52(x+y)z^2$ , International journal of Multidisciplinary research Volume 8, Issue 2, Page No.266-273, February 2022.
- [2] J.Shanthi, M.A.Gopalan, "On Non-Homogeneous cubic Equation with Four unknowns  $x^2 + y^2 + 4(35z^2 - 4 - 35w^2) = 6xyz$ ", Bio science Bio Technology Research Communication, Volume 14, Issue 05, 126- 129, March2021.
- [3] J.Shanthi, M.A.Gopalan, "A Search on Non-Distinct Integer Solutions to Cubic Diophantine Equation with four unknowns  $x^2 - xy + y^2 + 4w^2 = 8z^3$ , International Research Journal of Education and Technology, Volume 2 Issue 01, Page No:27-32, May 2021.
- [4] J.Shanthi, M.A.Gopalan, "On the non-homogeneous cubic Diophantine equation with four unknowns  $x^2 + y^2 + 4((2k^2 - 2k)z^2 - 4 - w^2) = (2k^2 - 2k + 1)xyz$ ", International journal of Mathematics and Computing Techiques, Volume 4, Issue 3, Page No:01-05, June 2021.
- [5] R.Sathiyapriya, N.Thiruniraiselvi, M.A.Gopalan, A Modish Glance of Integer Solutions to Non homogeneous Cubic Diophantine Equation with Three Unknowns  $5(x^2 + y^2) = 13z^3$ , Nanotechnology Perceptions, 21(1), (2025), 275-280, 2025.
- [6] K.Meena, M.A.Gopalan, S.Vidhyalakshmi and J.Shanthi, Observations on the Quartic equation with Six unknowns  $(x^3 - y^3)z = (w^2 - p^2)R^2$ , Global Journal of Pure and Applied Mathematics, Vol. 1, No. 3, 1435-1444, 2015.
- [7] M. A. Gopalan, S. Vidhyalakshmi, J. Shanthi, "The non-homogeneous quintic equations with five unknowns  $x^4 - y^4 + 2k(x^2 + y^2)(x - y + k) = (a^2 + b^2)(z^2 - w^2)p^{3n}$ ", open journal of applied and theoretical mathematics, vol-2, no 3, 08-13, sep 2016.
- [8] J.Shanthi, S.Vidhyalakshmi, M.A.Gopalan, On Finding Integer Solutions to Sextic Equation with Three Unknowns  $x^2 + y^2 = 8z^6$ , The Ciencia and Engenharia-Science and Engineering Journal, 11(1), 350-355, 2023.
- [9] J.Shanthi, M.A.Gopalan, On Non-homogeneous Sextic Equation with Three Unknowns  $y^2 + 3x^2 = 16z^6$ , IJRPR, 4(7), 1984-1988, 2023.

- [10] M.A.Gopalan, S. Vidhyalakshmi and J.Shanthi, Observations on the non-homogeneous sextic equation with five unknowns  $2(x^2 - y^2)(x^2 + y^2 - xy) = 7(z^2 - w^2)p^4$ , American International Journal of Research in Science, Technology, Engineering and Mathematics, Vol. 15, Issue 2, 169-172, June-August 2016.
- [11] J.Shanthi, N.Thiruniraiselvi and M.A.Gopalan, Patterns of Integer solutions to Non-homogeneous Ternary Heptic Diophantine Equation  $x^2 + y^2 = (a^2 + b^2)z^7$ , Gongcheng Kexue Xuebao, 11(3), 41-47, 2026.
- [12] M.A.Gopalan, S.Vidhyalakshmi and J.Shanthi, On the non-homogeneous Octic Equation with four unknowns  $x^2 = y^3 + z^5w^3$ , International Journal of Advanced and Innovative Research, Vol.5, Issue1, 13-15, 2015.
- [13] N.Thiruniraiselvi, M.A.Gopalan, A Search on Integer Solutions to Ternary Non-homogeneous Nonic Diophantine Equation  $\alpha(x^2 + y^2) - (2\alpha - 1)xy = 4\alpha z^9$ , Advances in Nonlinear Variational Inequalities, 27(2), 13-17, 2024.
- [14] M.A.Gopalan, S.Vidhyalakshmi and J.Shanthi, On the Diophantine equation of Degree Ten with Six Unknowns  $3(x^2 - y^2)^3 + 4T^6P^2(x^2 - y^2) = (z^4 - w^4)p^2$ , International Journal of Innovative Science, Engineering and Technology, Vol.3, Issue 5, 397-399, May 2016.
- [15] J.Shanthi, N.Thiruniraiselvi and M.A.Gopalan, Techniques to solve Diophantine Equation of Degree Ten with Six Unknowns  $3(x^2 - y^2)^3 + 4(a^2 + b^2)(x^2 - y^2)P^2T^6 = (z^4 - w^4)P^2$ , IRJMETS, Vol.7, Issue 6, 73-77, June 2025.
- [16] N.Thiruniraiselvi, M.A.Gopalan, A Portrayal of Integer Solutions to Non-homogeneous Quinary Nonic Equation  $x^4 - y^4 = 10(z^2 - w^2)P^7$ , Boletim da Sociedade Paranaense de Matemática, (44) 12 : 1-6, 2026.
- [17] N.Thiruniraiselvi, M.A.Gopalan, A Search on Integer Solutions to Ternary Non-homogeneous Nonic Diophantine Equation  $\alpha(x^2 + y^2) - (2\alpha - 1)xy = 4\alpha z^9$ , "Advances in Nonlinear Variational Inequalities", 27(2), 13-17, 2024.
- [18] N.Thiruniraiselvi, M.A.Gopalan, A Collection of Integer Solutions to Non-homogeneous Ternary Higher Degree Diophantine Equation  $3(x^2 + y^2) - 5xy = (k^2 + 11)z^{2s+1}$ , Boletim da Sociedade Paranaense de Matemática, (44) 10 : 1-8, 2026.
- [19] N.Thiruniraiselvi, M.A.Gopalan, Techniques to solve Diophantine Equation of Degree Ten with Six Unknowns  $x^6 - y^6 - 3456z^3 = 800(p^2 - q^2)R^8$ , IJO International Journal of Mathematics, Vol 7(5), Pg 1-7, 2024.



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